EE103 Lect #6 Oct 11, 2017

Note: Course materials (lecture slides, HWs & solutions) are posted on EE103 website:
https://ee103-fall2017-01.courses.soe.ucsc.edu/

also webcast can be watched at
https://webcast.ucsc.edu & EE103

Causality of System $x(t) \rightarrow \boxed{H} \rightarrow y(t)$

H is causal (= nonanticipatory) if $y(t_0)$ for any t_0 depends only on $x(t), t \le t_0$.

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convolution of $f_1(t), f_2(t)$

Notation $f_1(t) * f_2(t)$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

$$\underset{\tau \leftarrow (t-\tau)}{=} \int_{-\infty}^{\infty} f_1(t-\tau)\, f_2(\tau)\, d\tau$$

$$= f_2(t) * f_1(t)$$ commutative

$x(t) \rightarrow \boxed{H} \rightarrow y(t)$

$\delta(t) \xrightarrow{H} h(t)$

impulse input — impulse response

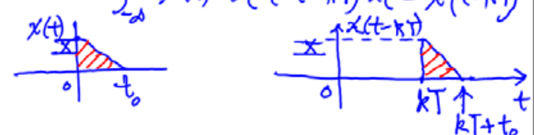
- $\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, d\tau = 1$
- $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\, dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\, dt = x(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)\, dt = x(t_0)$

also $x(t)\,\delta(t-\tau) = x(\tau)\,\delta(t-\tau)$

- $x(t)*\delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\, d\tau = x(t)\int_{-\infty}^{\infty}\delta(t-\tau)\, d\tau = x(t)$

$x(t) * \delta(t-kT)$

$$= \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau-kT)\, d\tau = x(t-kT)$$

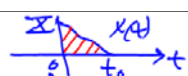


$$x(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT) = ?$$

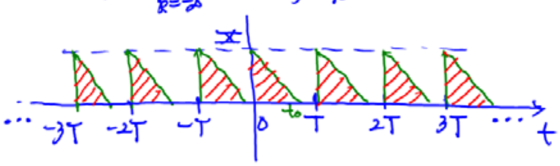
$$\Rightarrow \int_{-\infty}^{\infty} x(\tau) \sum_{k=-\infty}^{\infty} \delta(t-\tau-kT)\, d\tau = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau-kT)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-kT)\,\delta(t-\tau-kT)\, d\tau = \sum_{k=-\infty}^{\infty} x(t-kT)$$

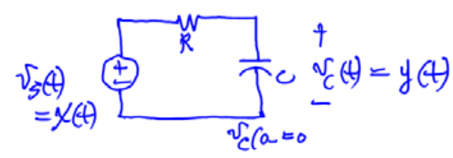
For $t_0 < T$,



$x(t) * \sum_{k=-\infty}^{\infty}\delta(t-kT)$ is



Let us consider a simple RC circuit



$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dy(t)}{dt} \quad (1)$$

$$i_C(t) = i_R(t) = \frac{v_s(t) - v_C(t)}{R} = \frac{x(t)-y(t)}{R} \quad (2)$$

From (1) & (2)

$$C\frac{dy}{dt} = \frac{1}{R}(x-y) \text{ or } \boxed{RC\frac{dy}{dt} = x-y} \quad (3)$$

From (3) in Laplace Transformed domain
(which will be covered in detail later on),

$$RC[sY(s) - y(0_-)] = X(s) - Y(s) \quad (4)$$

with $y(0_-) = v_C(0_-) = 0$ (by assumption).

$$(4) \rightarrow [RCs+1]\,Y(s) = X(s)$$

$$Y(s) = \frac{1}{RCs+1}X(s) \quad (5)$$

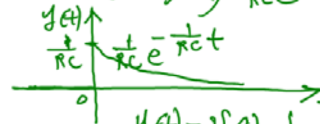
When $x(t) = \delta(t)$ impulse function

$$X(s) = \int_{-\infty}^{\infty}\delta(t)e^{-st}dt = 1, \text{ and}$$

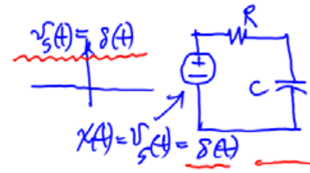
$$(5) \rightarrow Y(s) = \frac{1}{RCs+1} \quad (6)$$

Taking Inverse Laplace Transform of (6) →

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{RCs+1}\right) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

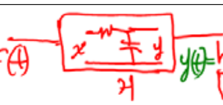


$$y(t) = v_C(t) = \begin{cases} 0 & t < 0 \\ \frac{e^{-\frac{1}{RC}t}}{RC} & t \geq 0 \end{cases}$$



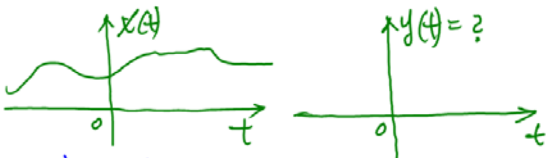
$$y(t) = \frac{1}{RC}u(t)e^{-\frac{1}{RC}t}$$

Impulse Response $h(t)$



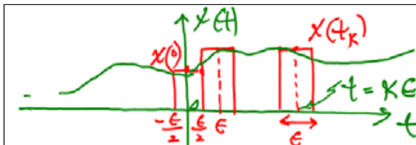
$x(t) = \delta(t)$ → $\mathcal{H}$ → $y(t) = h(t) = \frac{1}{RC}u(t)e^{-\frac{1}{RC}t}$ Impulse Response

For arbitrary $x(t)$ what would be $y(t)$=?



(Ans) $y(t) = x(t) * h(t)$

WHY?



We no that $x(t)$ can be expressed as

$$x(t) = \lim_{\epsilon \to 0}\sum_{k=-\infty}^{\infty} x(k\epsilon)\frac{1}{\epsilon}\mathrm{rect}(t-k\epsilon|\epsilon)\,\epsilon$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Also $y(t) = \mathcal{H}\,x(t)$

$$= \mathcal{H}\left[\lim_{\epsilon \to 0}\sum_{k=-\infty}^{\infty} x(k\epsilon)\,\mathrm{rect}(t-k\epsilon|\epsilon)\right]$$

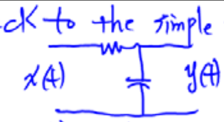
$$= \lim_{\epsilon \to 0}\sum_{k=-\infty}^{\infty} x(k\epsilon)\,\mathcal{H}\left[\frac{1}{\epsilon}\mathrm{rect}(t-k\epsilon|\epsilon)\right]\epsilon$$

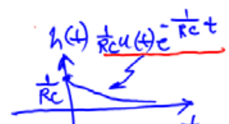
$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= x(t) * h(t) = h(t) * x(t)$$

convolution

Back to the simple RC circuit

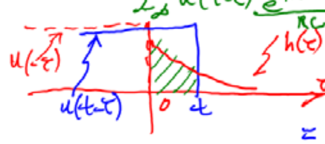




what is $y(t)$ for $x(t) = u(t)$?

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)\frac{e^{-\frac{1}{RC}(t-\tau)}}{RC}u(t-\tau)\,d\tau$$

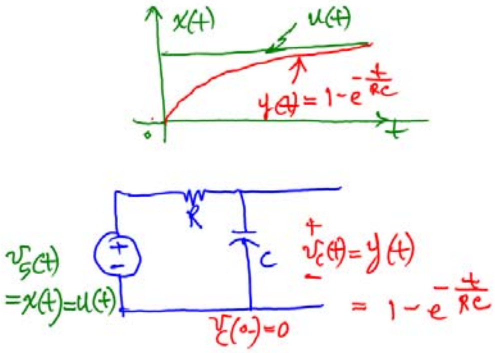
$$= \int_{-\infty}^{\infty} u(t-\tau)\frac{e^{-\frac{1}{RC}\tau}}{RC}u(\tau)\,d\tau$$



$$= \int_0^t \frac{1}{RC}e^{-\frac{1}{RC}\tau}\,d\tau$$

$$= \int_0^t d\left[-e^{-\frac{1}{RC}\tau}\right]$$

$$= \left(1 - e^{-\frac{1}{RC}t}\right) = y(t)$$



Convolution of periodic signal

2.7. Let $h(t)$ be the triangular pulse shown in Fig. 2-10(a) and let $x(t)$ be the unit impulse train [Fig. 2-10(b)] expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.68}$$

Determine and sketch $y(t) = h(t) * x(t)$ for the following values of T: (a) $T = 3$, (b) $T = 2$, (c) $T = 1.5$.

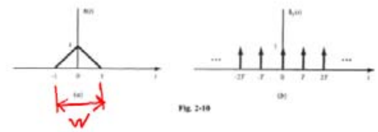


Fig. 2-10

$$y(t) = h(t) * \delta_T(t) = h(t) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT)\right]$$
$$= \sum_{n=-\infty}^{\infty} h(t) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} h(t - nT) \tag{2.69}$$

(a) For $T = 3$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 3n) \qquad T > W$$

which is sketched in Fig. 2-11(a).

(b) For $T = 2$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 2n) \qquad T = W$$

which is sketched in Fig. 2-11(b).

(c) For $T = 1.5$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 1.5n) \qquad T < W$$

which is sketched in Fig. 2-11(c). Note that when $T < 2$, the triangular pulses are no longer separated and they overlap.

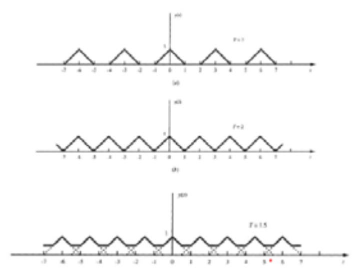


Fig. 2-11